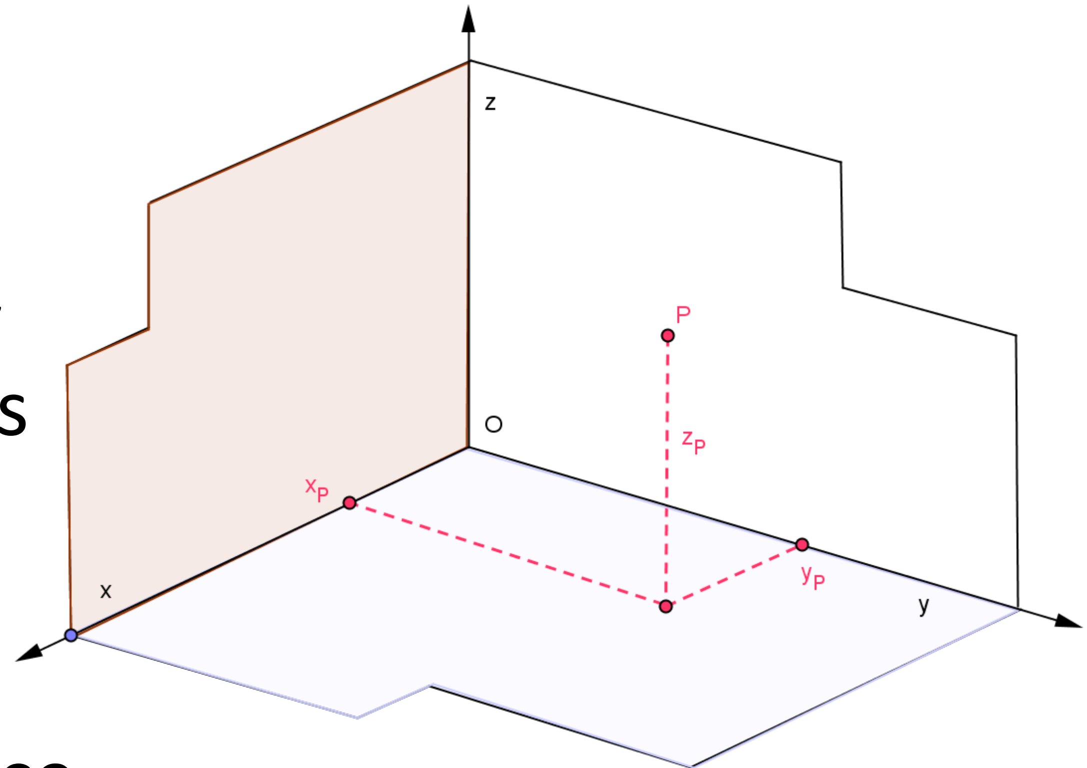
Vectors and operations with vectors

Orthogonal Cartesian coordinate system

- origin O
- 3 coordinate axes

  $x = o_x, y = o_y, z = o_z$
- 3 coordinate planes

  Oxy, Oxz, Oyz



Each point in the space

is identified with an order triple of real numbers

Cartesian coordinates $P = [x_P, y_P, z_P]$

Euclidean distance of two points

$P_1 = [x_1, y_1, z_1]$ and $P_2 = [x_2, y_2, z_2]$

is determined by formula

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Vector in E^3 is determined by 2 points

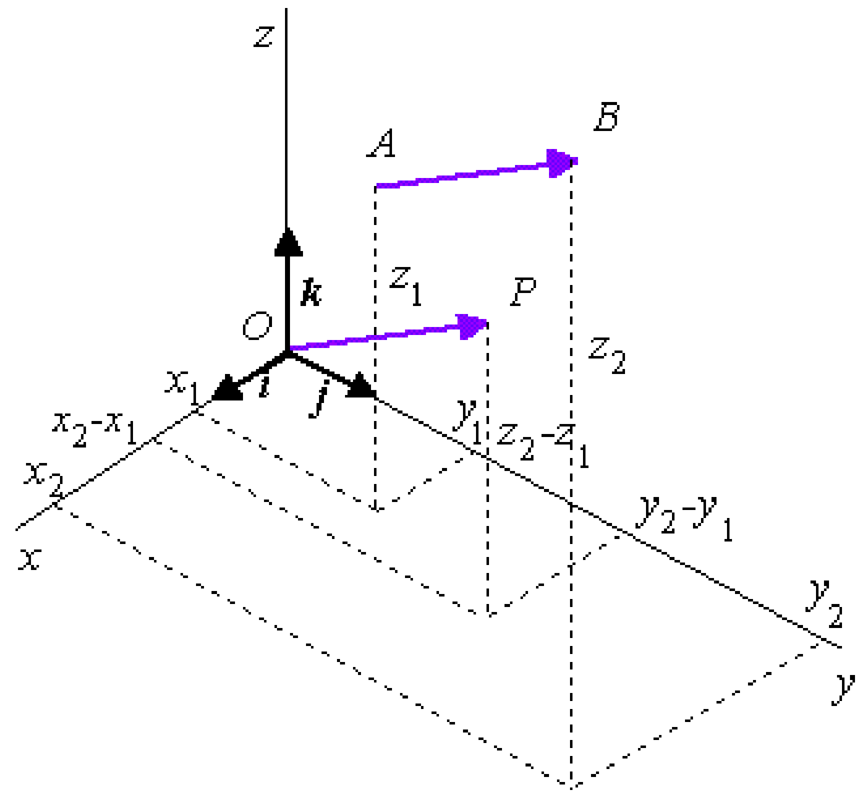
$$\mathbf{a} = \overrightarrow{P_1 P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

Position vector of point P is vector

$$\vec{\mathbf{a}} = \overrightarrow{OP} = (x_P, y_P, z_P)$$

Vector is any oriented line segment

$$\vec{\mathbf{v}} = \overrightarrow{AB}$$



A – start point

B – end point

of one vector location

Coordinates of vector

$$A = [x_A, y_A, z_A]$$

$$B = [x_B, y_B, z_B]$$

$$\vec{\mathbf{v}} = B - A = (x_B - x_A, y_B - y_A, z_B - z_A)$$

Norm – length of vector

$$\vec{\mathbf{v}} = \vec{AB} = B - A = (x_B - x_A, y_B - y_A, z_B - z_A) = (v_1, v_2, v_3)$$

is the distance of its determining points *A* and *B*

$$\left|\vec{\mathbf{v}}\right| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2} = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

$\left|\vec{\mathbf{v}}\right| = 1$ - unit vector

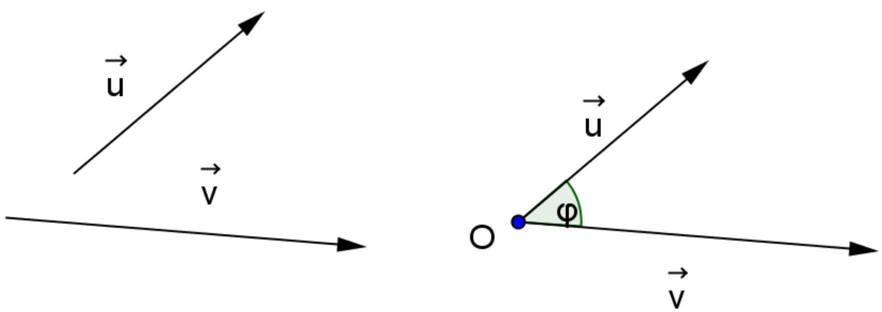
$\vec{\mathbf{0}} = (0,0,0)$ - zero vector

Angle of 2 vectors

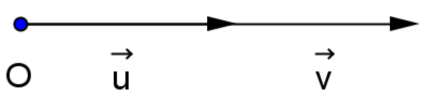
$$\vec{\mathbf{u}}, \vec{\mathbf{v}} \qquad \varphi = \angle(\vec{\mathbf{u}}, \vec{\mathbf{v}})$$

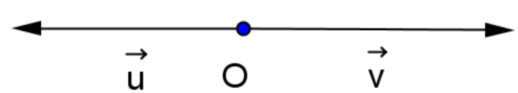
$$\left|\vec{\mathbf{u}}\right| \neq 0,\ \left|\vec{\mathbf{v}}\right| \neq 0,\ \varphi \in \langle 0, \pi \rangle$$

non-parallel vectors



$$\varphi \in (0, \pi)$$



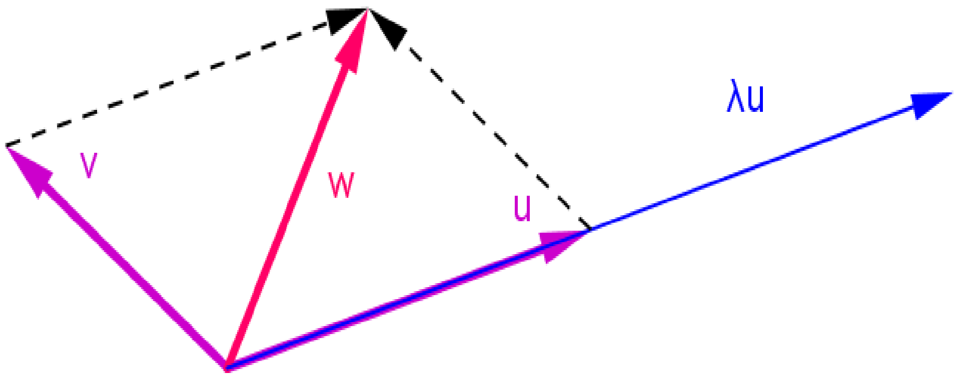


$$\varphi = 0 \qquad\qquad \varphi = \pi$$

parallel (collinear) vectors

Operations with vectors

Sum of vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$



$$\vec{\mathbf{u}} = (u_1, u_2, u_3)$$

$$\vec{\mathbf{v}} = (v_1, v_2, v_3)$$

$$\vec{\mathbf{w}} = \vec{\mathbf{u}} + \vec{\mathbf{v}} =$$

$$= (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

Scalar multiple of a vector

$$\vec{\mathbf{u}} = (u_1, u_2, u_3)$$

$$\vec{\mathbf{u}} \,||\, \vec{\mathbf{v}} \Leftrightarrow \exists \lambda \in R, \vec{\mathbf{u}} = \lambda \vec{\mathbf{v}}$$

$$\lambda \vec{\mathbf{u}} = (\lambda u_1, \lambda u_2, \lambda u_3)$$

Vectors $\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}$ are linearly dependent, if there exists linear combination of these vectors

$$k.\vec{\mathbf{a}}+l.\vec{\mathbf{b}}+m.\vec{\mathbf{c}}=\vec{\mathbf{0}}$$

such that at least one from coefficients *k*, *l*, *m* is a nonzero real number, i.e. $k^2+l^2+m^2\neq 0$

This means that at least one from vectors $\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}$ is a linear combination of the two others.

Unit vectors $\vec{\mathbf{i}}=(1,0,0),\vec{\mathbf{j}}=(0,1,0),\vec{\mathbf{k}}=(0,0,1)$ form ortho-normal basis of the three dimensional space.

Scalar product of vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$

$$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = \left|\vec{\mathbf{u}}\right| \cdot \left|\vec{\mathbf{v}}\right| \cos\varphi, \quad \varphi = \angle\left(\vec{\mathbf{u}}, \vec{\mathbf{v}}\right)$$

$$\text{If } \vec{\mathbf{u}} = \vec{\mathbf{0}} \vee \vec{\mathbf{v}} = \vec{\mathbf{0}} \Rightarrow \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$$

$$\vec{\mathbf{u}} = (u_1, u_2, u_3),\ \vec{\mathbf{v}} = (v_1, v_2, v_3),\ \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$$\cos\varphi = \frac{\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}}{\left|\vec{\mathbf{u}}\right| \cdot \left|\vec{\mathbf{v}}\right|}, \quad \varphi = \angle\left(\vec{\mathbf{u}}, \vec{\mathbf{v}}\right), \quad \vec{\mathbf{u}} \neq \vec{\mathbf{0}} \wedge \vec{\mathbf{v}} \neq \vec{\mathbf{0}}$$

$$\vec{\mathbf{u}} \perp \vec{\mathbf{v}} \Leftrightarrow \vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0, \quad \vec{\mathbf{u}} \neq \vec{\mathbf{0}} \wedge \vec{\mathbf{v}} \neq \vec{\mathbf{0}}$$

Vector product of vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ is vector $\vec{\mathbf{w}} = \vec{\mathbf{u}} \times \vec{\mathbf{v}}$

1. $\left|\vec{\mathbf{w}}\right| = \left|\vec{\mathbf{u}}\right| . \left|\vec{\mathbf{v}}\right| \sin\varphi, \quad \varphi = \angle\left(\vec{\mathbf{u}}, \vec{\mathbf{v}}\right)$

2. $\vec{\mathbf{w}} \perp \vec{\mathbf{u}} \wedge \vec{\mathbf{w}} \perp \vec{\mathbf{v}}$

3. Vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$ form right-handed system.

$$\vec{\mathbf{u}} = (u_1, u_2, u_3) \qquad \vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\vec{\mathbf{v}} = (v_1, v_2, v_3)$$

$$\vec{\mathbf{u}} \parallel \vec{\mathbf{v}} \Leftrightarrow \vec{\mathbf{u}} \times \vec{\mathbf{v}} = \vec{\mathbf{0}}, \quad \vec{\mathbf{u}} \neq \vec{\mathbf{0}} \wedge \vec{\mathbf{v}} \neq \vec{\mathbf{0}}$$

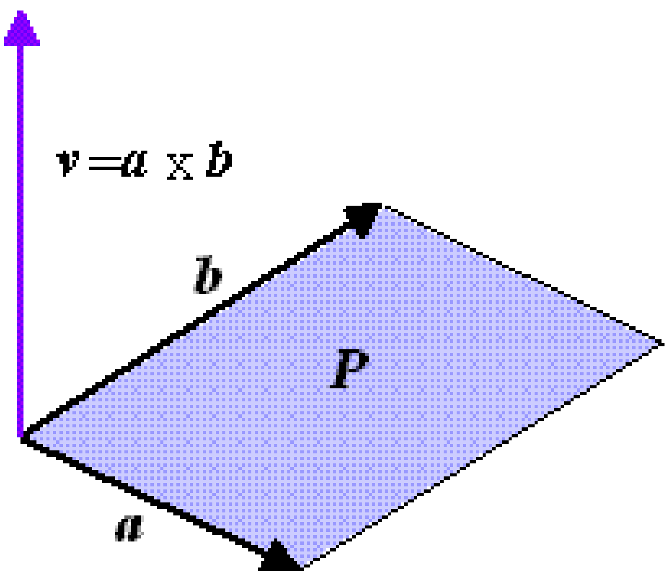
Geometric interpretation of vector product

Length of vector $\vec{\mathbf{v}} = \vec{\mathbf{a}} \times \vec{\mathbf{b}}$

equals to the area *P* of a parallelogram with sides

in non-collinear vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}$

$$P = \left| \vec{\mathbf{a}} \times \vec{\mathbf{b}} \right|$$



Mixed triple product of vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$ is number

$$\left[\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}\right] = \left(\vec{\mathbf{u}} \times \vec{\mathbf{v}}\right) . \vec{\mathbf{w}}$$

$$\vec{\mathbf{u}} = (u_1, u_2, u_3)$$

$$\vec{\mathbf{v}} = (v_1, v_2, v_3)$$

$$\vec{\mathbf{w}} = (w_1, w_2, w_3)$$

$$\left[\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}\right] = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}$ are in one plane (are coplanar)

if and only if $\left[\vec{\mathbf{u}}, \vec{\mathbf{v}}, \vec{\mathbf{w}}\right] = 0$

Geometric interpretation of mixed product

Absolute value of mixed scalar product of 3 vectors $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$ equals to the volume *V* of parallelepiped with edges in these non-coplanar vectors .

$$V = \left| \left[\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}} \right] \right|$$